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STATISTICAL MECHANICS ON A COMPACT SET WITH $\mathbf{Z}^\nu$ ACTION SATISFYING EXPANSIVENESS AND SPECIFICATION

BY
DAVID RUELLE

ABSTRACT. We consider a compact set Ω with a homeomorphism (or more generally a $\mathbf{Z}^\nu$ action) such that expansiveness and Bowen's specification condition hold. The entropy is a function on invariant probability measures. The pressure (a concept borrowed from statistical mechanics) is defined as function on $\mathcal{C}(\Omega)$—the real continuous functions on Ω. The entropy and pressure are shown to be dual in a certain sense, and this duality is investigated.

0. Introduction. Invariant measures for an Anosov diffeomorphism have been studied by Sinai [16], [17]. More generally, Bowen [2], [3] has considered invariant measures on basic sets for an Axiom A diffeomorphism. The problems encountered are strongly reminiscent of those of statistical mechanics (for a classical lattice system—see [14, Chapter 7]). In fact Sinai [18] has explicitly used techniques of statistical mechanics to show that an Anosov diffeomorphism does not in general have a smooth invariant measure.

In this paper, we rewrite a part of the general theory of statistical mechanics for the case of a compact set Ω satisfying expansiveness and the specification property of Bowen [2]. Instead of a $\mathbf{Z}$ action we consider a $\mathbf{Z}^\nu$ action as is usual in lattice statistical mechanics, where $\Omega = F^{\mathbf{Z}^\nu}$ (F: a finite set). This rewriting gives a more general and intrinsic formulation of (part of) statistical mechanics; it presents a number of technical problems, but the basic ideas are contained in the papers of Gallavotti, Lanford, Miracle, Robinson, and Ruelle [7], [11], [12], [13], etc. The ideas of Bowen [2] and Goodwyn [8] on the relation between topological and measure-theoretical entropy are also used.

We describe now some of our results in the case of a homeomorphism T of a metrizable compact set Ω satisfying expansiveness and specification (see §1).

Let $\Pi_a = \{x \in \Omega: T^a x = \{x\}\}$, and let $\mathcal{C}(\Omega)$ be the Banach space of real continuous functions on Ω. The *pressure* P is a continuous convex function on $\mathcal{C}(\Omega)$ defined by

$$P(\varphi) = \lim_{a\to\infty} \frac{1}{a} \log Z(\varphi, a), \qquad Z(\varphi, a) = \sum_{x\in\Pi_a} \exp \sum_{m=1}^{a} \varphi(T^m x)$$

(§2). Let I be the set of probability measures on Ω, invariant under T with the vague topology. The (measure theoretic) *entropy* s is an affine upper semicontinuous function on I defined in the usual way (§4). The following variational principle holds (§5)

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(0.1) $$P(\varphi) = \max_{\mu \in I}[s(\mu) + \mu(\varphi)], \qquad s(\mu) = \inf_{\varphi \in \mathcal{C}(\Omega)}[P(\varphi) - \mu(\varphi)].$$

Those μ for which the maximum is reached in (0.1) form a nonempty set I_φ. I_φ is a Choquet simplex and consists of precisely those $\mu \in I$ such that

$$P(\varphi + \psi) - P(\varphi) \geq \mu(\psi), \quad \text{all } \psi \in \mathcal{C}(\Omega).$$

Let $\mu_{\varphi,a}$ be the measure on Ω which is carried by Π_a and gives $x \in \Pi_a$ the mass

$$\mu_{\varphi,a}(\{x\}) = Z(\varphi, a)^{-1} \exp \sum_{m=1}^{a} \varphi(T^m x).$$

Then, any limit point of $\mu_{\varphi,a}$ as $a \to \infty$ is in I_φ (§3). There is a residual subset D of $\mathcal{C}(\Omega)$ such that I_φ consists of one single point μ_φ if $\varphi \in D$. In that case $\lim_{a\to\infty} \mu_{\varphi,a} = \mu_\varphi$.

Miscellaneous properties of invariant states are reviewed in §6.

I am indebted to J. Robbin for acquainting me with Bowen's papers, starting the present work.

1. **Notation and assumptions.** We denote by $|S|$ the cardinal of the set S. If $m = (m_1, \ldots, m_\nu) \in \mathbf{Z}^\nu$, $\nu \geq 1$, we let $\|m\| = \sup_i |m_i|$. Given integers $a_1, \ldots, a_\nu > 0$, we define $\Lambda(a) = \{m \in \mathbf{Z}^\nu\colon 0 \leq m_i < a_i\}$. If (Λ_α) is a directed family of finite subsets of $\mathbf{Z}^\nu$, $\Lambda_\alpha \uparrow \infty$ means $|\Lambda_\alpha| \to \infty$ and $|\Lambda_\alpha + F|/|\Lambda_\alpha| \to 1$ for every finite $F \subset \mathbf{Z}^\nu$. In particular $\Lambda(a) \uparrow \infty$ when $a \to \infty$ (i.e. when $a_1, \ldots, a_\nu \to \infty$).

Let $\mathbf{Z}^\nu$ act by homeomorphisms on the compact set Ω. We suppose that Ω is metrizable with metric d. $\mathcal{C}(\Omega)$ is the space of real continuous functions on Ω with the sup norm. On the space $\mathcal{C}(\Omega)^*$ of real measures on Ω, we put the vague topology. We denote by δ_x the unit mass at x.

The following assumptions are made.(1)

1.1. *Expansiveness.* There exists $\delta^* > 0$ such that

$$(d(mx, my) \leq \delta^* \text{ for all } m \in \mathbf{Z}^\nu) \Rightarrow (x = y).$$

1.2. *Weak specification.* Given $\delta > 0$ there exists $p(\delta) > 0$ such that for any families $(\Lambda_i)_{i \in \mathcal{I}}$, $(x_i)_{i \in \mathcal{I}}$ satisfying

(i) if $i \neq j$, the distance of Λ_i, Λ_j (as subsets of $\mathbf{Z}^\nu$, with the distance $\|\cdot\|$) is $> p(\delta)$,

there is $x \in X$ such that

$$d(m_i x, m_i x_i) < \delta, \quad \text{all } i \in \mathcal{I}, \text{ all } m_i \in \Lambda_i.$$

1.3. *Strong specification.* Let $\mathbf{Z}^\nu(a)$ be the subgroup of $\mathbf{Z}^\nu$ with generators $(a_1, 0, \ldots, 0), \ldots, (0, \ldots, a_\nu)$, and let $\Pi_a = \{x \in \Omega\colon \mathbf{Z}^\nu(a)x = \{x\}\}$. For any

(1) Cf. Bowen [2].

families $(\Lambda_i)_{i\in\mathcal{I}}$, $(x_i)_{i\in\mathcal{I}}$ satisfying

$$\text{(ii)}\qquad \begin{array}{l}\Lambda_i \subset \Lambda(a) \text{ for all } i \text{ and, if } i \neq j,\\ \text{the distance of } \Lambda_i + \mathbf{Z}^\nu(a) \text{ and } \Lambda_j \text{ is} > p(\delta),\end{array}$$

there is $x \in \Pi_a$ such that

$$d(m_i x, m_i x_i) < \delta, \quad \text{all } i \in \mathcal{I}, \text{ all } m_i \in \Lambda_i.$$

It is easily seen that strong specification implies weak specification. If Ω is a basic set for an Axiom A diffeomorphism ($\nu = 1$), it is known that expansiveness [19] holds, and that (strong) specification [2] holds for some iterate of the diffeomorphism.

We note that expansiveness has the following easy consequence.

1.4. **Proposition** [9]. *If* $0 < \delta$ *there exists* $q(\delta)$ *such that* $(d(mx,my) \leq \delta^*$ *if* $|m| < q(\delta)) \Rightarrow (d(x,y) < \delta)$.

2. Partition functions and pressure.

2.1. **Definitions.** Let $\delta > 0$; $E \subset \Omega$ is (δ, Λ)-separated if $(x,y \in E$, and $d(mx,my) < \delta$ for all $m \in \Lambda) \Rightarrow (x = y)$. Let $\varphi \in \mathcal{C}(\Omega)$. Given $\delta > 0$ and a finite $\Lambda \subset \mathbf{Z}^\nu$, or given $a = (a_1, \ldots, a_\nu)$ we introduce the *partition functions*

$$(2.1)\qquad Z(\varphi, \delta, \Lambda) = \max_E \sum_{x\in E} \exp \sum_{m\in\Lambda} \varphi(mx)$$

where the max is taken over all (δ, Λ)-separated sets, or

$$(2.2)\qquad Z(\varphi, a) = \sum_{x\in\Pi_a} \exp \sum_{m\in\Lambda(a)} \varphi(mx).$$

We write

$$(2.3)\qquad P(\varphi, \delta, \Lambda) = (1/|\Lambda|)\log Z(\varphi, \delta, \Lambda),$$

$$(2.4)\qquad P(\varphi, a) = (1/|\Lambda(a)|)\log Z(\varphi, a).$$

2.2. **Theorem.** *If* $0 < \delta < \delta^*$, *the following limits exist*:

$$(2.5)\qquad \lim_{\Lambda\uparrow\infty} P(\varphi, \delta, \Lambda) = P(\varphi),$$

$$(2.6)\qquad \lim_{a\to\infty} P(\varphi, a) = P(\varphi),$$

and define a finite-valued convex function P *on* $\mathcal{C}(X)$. *Furthermore*

$$(2.7)\qquad |P(\varphi) - P(\psi)| \leq \|\varphi - \psi\|$$

and if $\tau_m \psi(x) = \psi(mx)$, $t \in \mathbf{R}$,

$$(2.8)\qquad P(\varphi + \tau_m\psi - \psi + t) = P(\varphi) + t.$$

P is called the pressure.

Let $\epsilon > 0$; we choose $\delta' > 0$ so small that $\delta + 2\delta' \leq \delta^*$ and

$$(d(x,y) < \delta') \Rightarrow |\varphi(x) - \varphi(y)| < \epsilon; \tag{2.9}$$

then take $p(\delta')$ according to 1.2. Given a, write $b = (a_1 + p(\delta'), \dots, a_\nu + p(\delta'))$. We consider the partition $(\Lambda(b) + r)_{r \in \mathbf{Z}^\nu(b)}$ of $\mathbf{Z}^\nu$. For a finite $\Lambda \subset \mathbf{Z}^\nu$, let $R = \{r\colon \Lambda(b) + r \subset \Lambda\}$. Using specification we obtain

$$\begin{aligned} &Z(\varphi, \delta, \Lambda) \\ &\geq [Z(\varphi, \delta + 2\delta', \Lambda(a)) \exp(-|\Lambda(a)|\epsilon) \exp(-(|\Lambda(b)| - |\Lambda(a)|)\|\varphi\|)]^{|R|} \\ &\quad \cdot \exp(-(|\Lambda| - |R|\,|\Lambda(b)|)\|\varphi\|). \end{aligned} \tag{2.10}$$

Since Π_a is $(\delta^*, \Lambda(a))$-separated by expansiveness, we have also

$$Z(\varphi, \delta^*, \Lambda(a)) \geq Z(\varphi, a). \tag{2.11}$$

If $\Lambda \uparrow \infty$ we have $|R|\,|\Lambda(b)|/|\Lambda| \to 1$, and therefore (2.10) and (2.11) yield

$$\liminf_{\Lambda \uparrow \infty} P(\varphi, \delta, \Lambda) \geq \frac{|\Lambda(a)|}{|\Lambda(b)|} \cdot [P(\varphi, a) - \epsilon] - \left(1 - \frac{|\Lambda(a)|}{|\Lambda(b)|}\right)\|\varphi\|. \tag{2.12}$$

Suppose now that $\delta' < \frac{1}{2}\delta$, and let N be the cardinal of a finite cover of Ω by sets of diameter $< \delta$. Let F be a $(\delta', \Lambda(b))$ separated set such that

$$Z(\varphi, \delta', \Lambda(b)) = \sum_{y \in F} \exp \sum_{m \in \Lambda(b)} \varphi(my).$$

Given $x \in E$ and $r \in R$ we choose $y \in F$ such that $d((r + m)x, my) < \delta'$, for all $m \in \Lambda(b)$. The mapping $(x, r) \to y$ defines an injection $E \to F^R$, and therefore

$$Z(\varphi, \delta, \Lambda) \leq [Z(\varphi, \delta', \Lambda(b)) \exp(|\Lambda(b)|\epsilon)]^{|R|} (N \exp \|\varphi\|)^{|\Lambda| - |R|\,|\Lambda(b)|}. \tag{2.13}$$

Taking $c = (b_1 + p(\delta'), \dots, b_\nu + p(\delta'))$, strong specification gives

$$Z(\varphi, \delta', \Lambda(b)) \exp(-|\Lambda(b)|\epsilon) \exp(-(|\Lambda(c)| - |\Lambda(b)|)\|\varphi\|) \leq Z(\varphi, c). \tag{2.14}$$

From (2.13) and (2.14) we obtain

$$\limsup_{\Lambda \uparrow \infty} P(\varphi, \delta, \Lambda) \leq \frac{|\Lambda(c)|}{|\Lambda(b)|} P(\varphi, c) + 2\epsilon + \left(\frac{|\Lambda(c)|}{|\Lambda(b)|} - 1\right)\|\varphi\|. \tag{2.15}$$

Letting $a \to \infty$ in (2.12) and (2.15) we obtain (2.5) and (2.6).

The finiteness of $P(\varphi)$ follows from $\exp(-|\Lambda|\,\|\varphi\|) \leq Z(\varphi, \delta, \Lambda) \leq N^{|\Lambda|} \exp(|\Lambda|\,\|\varphi\|)$. The other properties follow from Lemma 2.3 below.

2.3. **Lemma.** *$P(\varphi, \delta, \Lambda)$ is a convex function of φ. Furthermore $|P(\varphi, \delta, \Lambda) - P(\psi, \delta, \Lambda)| \leq \|\varphi - \psi\|$ and $P(\varphi + t, \delta, \Lambda) = P(\varphi, \delta, \Lambda) + t$, if $t \in \mathbf{R}$. Similar* *properties hold for* $P(\varphi, a)$, *and also* $P(\varphi + \tau_m\psi - \psi, a) = P(\varphi, a)$.

We have $P(\varphi, \delta, \Lambda) = \max_E p(\varphi)$ where

$$p(\varphi) = (1/|\Lambda|)\log Z(\varphi), \qquad Z(\varphi) = \sum_{x\in E} \exp \sum_{m\in\Lambda} \varphi(mx),$$

$$\frac{d}{dt}p(\varphi + t\psi) = \frac{1}{Z(\varphi + t\psi)} \sum_x \frac{1}{|\Lambda|}\left[\sum_{m'} \psi(m'x)\right]\exp \sum_m [\varphi(mx) + t\psi(mx)].$$

Therefore

$$|\Lambda|\frac{d^2}{dt^2}p(\varphi + t\psi)|_{t=0}$$
$$= \frac{1}{Z^2}\sum_x \sum_y \frac{1}{2}\left[\sum_m \psi(mx) - \sum_m \psi(my)\right]^2 \exp \sum_m [\varphi(mx) + \varphi(my)] \geq 0.$$

On the other hand $|dp(\varphi + t\psi)/dt| \leq \|\psi\|$; hence

$$|p(\varphi) - p(\psi)| \leq \sup_{0\leq t\leq 1}\left|\frac{d}{dt}p(\varphi + t(\psi - \varphi))\right| \leq \|\psi - \varphi\|.$$

Finally $Z(\varphi + t) = e^{|\Lambda|t}Z(\varphi)$, $Z(\varphi + \tau_m\psi - \psi, a) = Z(\varphi, a)$.

2.4. **Remark.** Let Σ be the subgroup of $\mathbf{Z}^\nu$ with linearly independent generators $s_1, \ldots, s_\nu$, and define $\Lambda(\Sigma) = \{m \in \mathbf{Z}^\nu : m = \sum_1^\nu t_i s_i$ with t_i real, $0 \leq t_i < 1\}$. If a suitable extension of the strong specification property holds, one can prove

$$P(\varphi) = \lim_{\Lambda(\Sigma)\uparrow\infty} \frac{1}{|\Lambda(\Sigma)|} \log \sum_{x\in\Pi_\Sigma} \exp \sum_{m\in\Lambda(\Sigma)} \varphi(mx),$$

where $\Pi_\Sigma = \{x : \Sigma x = \{x\}\}$.

On the other hand, except for (2.6), Theorem 2.2 can be proved without the strong specification property (but assuming expansiveness and weak specification).

3. Equilibrium states.

3.1. **Definition.** Let $\mu_{\varphi,a}$ be the measure on Ω which is carried by Π_a and gives $x \in \Pi_a$ the mass

$$\mu_{\varphi,a}(\{x\}) = Z(\varphi, a)^{-1}\exp \sum_{m\in\Lambda(a)} \varphi(mx). \tag{3.1}$$

3.2. **Theorem.** (a) *Let* $I_\varphi \subset \mathcal{C}(\Omega)^*$ *be the set of measures* μ *such that*

$$P(\varphi + \psi) \geq P(\varphi) + \mu(\psi) \tag{3.2}$$

for all ψ (*equilibrium states for* φ). *Then* I_φ *is nonempty and there is a residual* (2) *set* $D \subset \mathcal{C}(\Omega)$ *such that* I_φ *consists of a single point* μ_φ *if* $\varphi \in D$.

(2) I.e. D is a countable intersection of dense open subsets of $\mathcal{C}(\Omega)$; in particular D is dense in $\mathcal{C}(\Omega)$ by Baire's theorem.

(b) I_φ *is convex, (vaguely) compact, and consists of* $\mathbf{Z}^\nu$ *invariant probability measures.*

(c) *The probability measure* $\mu_{\varphi,a}$ *is* $\mathbf{Z}^\nu$ *invariant, and*

$$\mu_{\varphi,a}(\psi) = dP(\varphi + t\psi, a)/dt\,|_{t=0}\,. \tag{3.3}$$

(d) *If* μ *is a (vague) limit point of the* $(\mu_{\varphi,a})$ *when* $a \to \infty$, *then* $\mu \in I_\varphi$. *In particular, if* $\varphi \in D$,

$$\lim_{a\to\infty} \mu_{\varphi,a} = \mu_\varphi\,. \tag{3.4}$$

(e) *If* $\mathcal{B}$ *is dense in* $\mathcal{C}(\Omega)$ *and is a separable Banach space with respect to a norm* $|||\cdot||| \geq \|\cdot\|$, *then* $D \cap \mathcal{B}$ *is residual in* $\mathcal{B}$.

(a) holds for any convex continuous function P on a separable Banach space (see Dunford-Schwartz [6, Theorem V.9.8]). This proves also (e).

Let μ satisfy (3.2). Then by (2.8),

$$0 = P(\varphi + \tau_m\psi - \psi) - P(\varphi) \geq \mu(\tau_m\psi - \psi) \geq -[P(\varphi - \tau_m\psi + \psi) - P(\varphi)] = 0$$

so that μ is $\mathbf{Z}^\nu$ invariant. Using (2.7) and (2.8) we obtain also $\pm\mu(\psi) \leq P(\varphi \pm \psi) - P(\varphi) \leq \|\psi\|$ and $\mu(1) = -\mu(-1) \geq -[P(\varphi - 1) - P(\varphi)] = 1$. Therefore $\|\mu\| \leq 1$, $\mu(1) \geq 1$ which implies that $\mu \geq 0$, $\|\mu\| = 1$, i.e. μ is a probability measure. Clearly, I_φ is convex and compact, and (b) is thus proved.

(c) follows readily from the definitions. From (3.3) and the convexity of $P(\cdot, a)$ (Lemma 2.3), we obtain

$$P(\varphi + \psi, a) \geq P(\varphi, a) + \mu_{\varphi,a}(\psi).$$

If $\mu_{\varphi,a} \to \mu$ this yields (3.2), proving (d).

4. Entropy.(3)

4.1. **Definitions.** Let $\mathcal{A} = (A_i)_{i\in\mathcal{I}}$ be a finite Borel partition of Ω, and Λ a finite subset of $\mathbf{Z}^\nu$. We denote by $\mathcal{A}^\Lambda$ the partition of Ω consisting of the sets $A(k) = \bigcap_{m\in\Lambda}(-m)A_{k(m)}$ indexed by maps $k\colon \Lambda \to \mathcal{I}$. We write

$$S(\mu, \mathcal{A}) = -\sum_i \mu(A_i)\log\mu(A_i). \tag{4.1}$$

Let I be the (convex compact) set of $\mathbf{Z}^\nu$-invariant probability measures on Ω.

4.2. **Theorem.** *If* $\mathcal{A}$ *consists of sets with diameter* $\leq \delta^*$, *and* $\mu \in I$, *then*

$$\lim_{\Lambda\uparrow\infty} \frac{1}{|\Lambda|} S(\mu, \mathcal{A}^\Lambda) = \inf_\Lambda \frac{1}{|\Lambda|} S(\mu, \mathcal{A}^\Lambda) = s(\mu). \tag{4.2}$$

(3) See also J.-P. Conze, *Entropie d'un groupe abélien de transformations* [Z. Wahrscheinlichkeitstheorie Verw. Gebiete **25** (1972), 11–30].

This limit is finite ≥ 0, and independent of $\mathcal{A}$. Furthermore, s is affine upper semicontinuous on I; s is called the entropy.

$S(\mu, \mathcal{A}^\Lambda)$ is an increasing function of Λ, and satisfies the strong subadditivity property

$$(4.3) \qquad S(\mu, \mathcal{A}^{\Lambda_1 \cup \Lambda_2}) + S(\mu, \mathcal{A}^{\Lambda_1 \cap \Lambda_2}) \leq S(\mu, \mathcal{A}^{\Lambda_1}) + S(\mu, \mathcal{A}^{\Lambda_2}).$$

[These are well-known properties. The increase follows from increase of the logarithm. To prove strong subadditivity we write $S(\mu, \mathcal{A}^\Lambda) = S_\Lambda$, and use the inequality $-\log(1/t) \leq t - 1$, then

$$\begin{aligned} &S_{\Lambda_1 \cup \Lambda_2} + S_{\Lambda_1 \cap \Lambda_2} - S_{\Lambda_1} - S_{\Lambda_2} \\ &\quad = - \sum_{k:\Lambda_1 \cap \Lambda_2 \to \Omega} \sum_{k':\Lambda_1 \backslash \Lambda_2 \to \Omega} \sum_{k'':\Lambda_2 \backslash \Lambda_1 \to \Omega} \mu(A(k,k',k'')) \log \frac{\mu(A(k,k',k''))\mu(A(k))}{\mu(A(k,k'))\mu(A(k,k''))} \\ &\quad \leq \sum_{kk'k''} \mu(A(k,k',k'')) \left[\frac{\mu(A(k,k'))\mu(A(k,k''))}{\mu(A(k,k',k''))\mu(A(k))} - 1 \right] \\ &\quad = \sum_{kk'} \frac{\mu(A(k,k'))}{\mu(A(k))} \sum_{k''} \mu(A(k,k'')) - \sum_{kk'k''} \mu(A(k,k',k'')) \\ &\quad = \sum_{kk'} \mu(A(k,k')) - 1 = 0.] \end{aligned}$$

If $\Lambda_1 \cap \Lambda_2 = \varnothing$, (4.3) becomes subadditivity: $S(\mu, \mathcal{A}^{\Lambda_1 \cup \Lambda_2}) \leq S(\mu, \mathcal{A}^{\Lambda_1}) + S(\mu, \mathcal{A}^{\Lambda_2})$. Since $\mu \in I$ we have also $S(\mu, \mathcal{A}^\Lambda) = S(\mu, \mathcal{A}^{\Lambda+m})$ and therefore(4)

$$(4.4) \qquad \lim_{a \to \infty} \frac{1}{|\Lambda(a)|} S(\mu, \mathcal{A}^{\Lambda(a)}) = \inf_a \frac{1}{|\Lambda(a)|} S(\mu, \mathcal{A}^{\Lambda(a)}) = s.$$

Given $\epsilon > 0$, choose a such that $|\Lambda(a)|^{-1} S(\mu, \mathcal{A}^{\Lambda(a)}) \leq s + \epsilon$. Consider the partition $(\Lambda(a) + r)_{r \in \mathbf{Z}^\nu(a)}$ of $\mathbf{Z}^\nu$, and let $R = \{r \in \mathbf{Z}^\nu(a) : (\Lambda(a) + r) \cap \Lambda \neq \varnothing\}$. If $\Lambda_+ = \bigcup_{r \in R} (\Lambda(a) + r)$ we have by increase and subadditivity

$$S(\mu, \mathcal{A}^\Lambda) \leq S(\mu, \mathcal{A}^{\Lambda_+}) \leq |R| S(\mu, \mathcal{A}^{\Lambda(a)}) \leq |R| \, |\Lambda(a)| (s + \epsilon).$$

But $|R| \, |\Lambda(a)|/|\Lambda| \to 1$ when $\Lambda \uparrow \infty$, and therefore

$$(4.5) \qquad \limsup_{\Lambda \uparrow \infty} |\Lambda|^{-1} S(\mu, \mathcal{A}^\Lambda) \leq s + \epsilon.$$

Strong subadditivity shows that

$$(4.6) \qquad S(\mu, \mathcal{A}^{\Lambda \cup \{m\}}) - S(\mu, \mathcal{A}^\Lambda) \geq S(\mu, \mathcal{A}^{\Lambda' \cup \{m\}}) - S(\mu, \mathcal{A}^{\Lambda'})$$

when $m \notin \Lambda' \supset \Lambda$. This permits an estimate of the increase in the entropy for a set Λ to which points are added successively in lexicographic order. In

(4) See for instance [14, Proposition 7.2.4].

particular if Λ is fixed and one takes for Λ' the sets successively obtained in the lexicographic construction of a large $\Lambda(a)$, (4.6) holds for most Λ'. Therefore

$$S(\mu, \mathscr{A}^{\Lambda \cup \{m\}}) - S(\mu, \mathscr{A}^{\Lambda}) \geq \lim_{a \to \infty} |\Lambda(a)|^{-1} S(\mu, \mathscr{A}^{\Lambda(a)}) = s$$

and hence

$$S(\mu, \mathscr{A}^{\Lambda}) \geq |\Lambda| s \tag{4.7}$$

for all Λ; (4.2) follows from (4.5) and (4.7).

Let $x \in \Omega$ and for each $m \in \Lambda$, let B_m be the union of those A_i which contain x in their closure. Then $B_\Lambda = \bigcap_{m \in \Lambda} (-m) B_m$ contains x in its interior and is a union of elements of $\mathscr{A}^\Lambda$. If $y \in B_\Lambda$ and $\Lambda = \{m\colon |m| < q(\delta)\}$, then $d(x,y) < \delta$ (see (1.4)). Therefore the σ-field generated by the $\mathscr{A}^\Lambda$ is the Borel σ-field. The Kolmogorov-Sinai theorem (see [20, 5.5]) holds for the group $\mathbf{Z}^\nu$ and implies that the limit (4.2) is independent of $\mathscr{A}$ (it is clearly finite ≥ 0).

If $\mu, \mu' \in I$, and $0 < \alpha < 1$, the following inequalities are standard:

$$\begin{aligned} \alpha S(\mu, \mathscr{A}) + (1-\alpha) S(\mu', \mathscr{A}) &\leq S(\alpha\mu + (1-\alpha)\mu', \mathscr{A}) \\ &\leq \alpha S(\mu, \mathscr{A}) + (1-\alpha) S(\mu', \mathscr{A}) + \log 2. \end{aligned} \tag{4.8}$$

[Writing $\mu_i = \mu(A_i)$, $\mu'_i = \mu(A'_i)$ we have indeed, using the convexity of $t \log t$ and the increase of $\log t$,

$$\begin{aligned} -\sum_i [\alpha\mu_i \log \mu_i &+ (1-\alpha)\mu'_i \log \mu'_i] \\ &\leq -\sum_i [\alpha\mu_i + (1-\alpha)\mu'_i] \log[\alpha\mu_i + (1-\alpha)\mu'_i] \\ &\leq -\sum_i [\alpha\mu_i \log \alpha\mu_i + (1-\alpha)\mu'_i \log(1-\alpha)\mu'_i] \\ &= -\sum_i [\alpha\mu_i \log \mu_i + (1-\alpha)\mu'_i \log \mu'_i] - \alpha \log \alpha - (1-\alpha)\log(1-\alpha) \\ &\leq -\sum_i [\alpha\mu_i \log \mu_i + (1-\alpha)\mu'_i \log \mu'_i] + \log 2.] \end{aligned}$$

(4.8) implies that s is affine.

To prove that s is upper semicontinuous at μ, choose $\mathscr{A}$ such that the boundaries of the A_i have μ-measure zero. [If $x \in \Omega$ one can choose $\delta \leq \frac{1}{2}\delta^*$ such that the boundary of the sphere of radius δ centered at x has μ-measure 0. Take a finite covering of Ω by such spheres and let $\mathscr{A}$ be generated by this covering.] The boundaries of the $A(k) \in \mathscr{A}^\Lambda$ have also measure 0, hence

$$\lim_{\mu' \to \mu} \mu'(A(k)) = \mu(A(k)), \qquad \lim_{\mu' \to \mu} S(\mu', \mathscr{A}^\Lambda) = S(\mu, \mathscr{A}^\Lambda),$$

and s is upper semicontinuous as inf of continuous functions.

4.3. Remarks. (a) Theorem 4.2 reduces to the usual definition of the measure theoretic entropy for $\nu = 1$.

(b) The condition that the diameters of the A_i are $\leq \delta^*$ can be replaced by the weaker condition that the $\mathcal{A}^\Lambda$ generate the Borel σ-field (see the proof).

(c) The proof of Theorem 4.2 assumes expansiveness, but specification is not used.

5. Variational principle.

5.1. **Theorem.** *For all* $\varphi \in \mathcal{C}(\Omega)$,

$$P(\varphi) = \max_{\mu \in I}[s(\mu) + \mu(\varphi)] \tag{5.1}$$

and the maximum is reached precisely on I_φ. *For all* $\mu \in I$,

$$s(\mu) = \inf_{\varphi}[P(\varphi) - \mu(\varphi)].$$

Let $\varphi \in \mathcal{C}(\Omega)$ and $\mu \in I$ be given. Since Ω is metrizable compact, there exists a finite set $\{\psi_1, \ldots, \psi_t\}$ of elements of $\mathcal{C}(\Omega)$ such that if $|\psi_l(x) - \psi_l(y)| < 1$ for $l = 1, \ldots, t$, then $d(x,y) < \delta^*$. Given $\epsilon > 0$ and a we construct a partition $\mathcal{B} = (B_i)_{i \in \mathcal{J}}$ consisting of sets of the form $B_i = \{x\colon u_{ilm} \leq \psi_l(mx) < v_{ilm}$ and $u'_{im} \leq \varphi(mx) < v'_{im}$ for all i, l, and $m \in \Lambda(a)\}$. By suitable choice of the u_{ilm}, v_{ilm}, u'_{im}, v'_{im} we can achieve that

(a) the diameter of each set $(-m)B_i$, for $m \in \Lambda(a)$, is $\leq \delta^*$;

(b) if B_i, B_j are adjacent (i.e. $\bar{B}_i \cap \bar{B}_j \neq \emptyset$) and $x \in B_i$, $y \in B_j$, then

$$|\varphi(mx) - \varphi(my)| < \epsilon/2 \text{ for all } m \in \Lambda(a);$$

(c) each $x \in X$ is contained in the closure of at most $(t + 1)|\Lambda(a)| + 1$ sets B_i. (5)

Because of (c) there exists δ, $0 < \delta < \delta^*$, such that for each x there are at most $(t + 1)|\Lambda(a)| + 1$ sets B_i with distance $< \delta$ to x, and these sets are all adjacent to that containing x.

Let R be a subset of $\mathbf{Z}^\nu(a)$, then

$$\inf_R \frac{1}{|R|} S(\mu, \mathcal{B}^R) = |\Lambda(a)| s(\mu). \tag{5.3}$$

To see this notice that the $\mathcal{B}^R$ generate the Borel σ-field (by (a) above), and apply Remark 4.3(b) with $\mathbf{Z}^\nu$ replaced by $\mathbf{Z}^\nu(a)$. It follows that the left-hand side of (5.3) is not changed if $\mathcal{B}$ is replaced by $\mathcal{A}^{\Lambda(a)}$, and (5.3) follows. If E is a maximal (δ, R)-separated set, for each $k\colon R \to \mathcal{J}$ such that $B(k) \neq \emptyset$, one can choose $x \in B(k)$ and then $x_k \in E$ such that $d(rx_k, rx) < \delta$, all $r \in R$. By the choice of δ, rx_k is in a set B_i adjacent to $B_{k(r)}$. Therefore, by (b),

$$\left| \sum_{m \in \Lambda(a)} \varphi((r + m)x_k) - \sum_{m \in \Lambda(a)} \varphi(my) \right| < |\Lambda(a)|\epsilon/2$$

(5) The B_i may be viewed as $(t + 1)|\Lambda(a)|$-dimensional rectangles and they can be adjusted so that at most $(t + 1)|\Lambda(a)| + 1$ meet at a corner. This idea is used by Goodwyn [8].

for all $y \in B_{k(r)}$. Choose $y_i \in B_i$ for each $i \in \mathcal{I}$, then

$$
\begin{aligned}
&\frac{1}{|R|} \sum_{k:R\to\mathcal{I}} \mu(B(k)) \sum_{r\in R} \sum_{m\in\Lambda(a)} \varphi((r+m)x_k) \\
&\qquad \geq \frac{1}{|R|} \sum_{r\in R} \sum_{i\in\mathcal{I}} \sum_{k:k(r)=i} \mu(B(k)) \left[\sum_{m\in\Lambda(a)} \varphi(my_i) - |\Lambda(a)|\epsilon/2 \right] \\
&\qquad = \frac{1}{|R|} \sum_{r\in R} \sum_{i\in\mathcal{I}} \mu(B_i) \sum_{m\in\Lambda(a)} \varphi(my_i) - |\Lambda(a)|\epsilon/2 \\
&\qquad = \sum_{i\in\mathcal{I}} \mu(B_i) \sum_{m\in\Lambda(a)} \varphi(my_i) - |\Lambda(a)|\epsilon/2 \\
&\qquad \geq |\Lambda(a)|(\mu(\varphi) - \epsilon).
\end{aligned} \tag{5.4}
$$

Notice that each $x_k \in E$ comes from at most $[(t+1)|\Lambda(a)|+1]^{|R|}$ different k's. Using this, and also (5.3), (5.4) and the concavity of the log, we obtain

$$
\begin{aligned}
&|\Lambda(a)|(s(\mu) + \mu(\varphi) - \epsilon) \\
&\qquad \leq \frac{1}{|R|} \sum_k \mu(B(k)) \left[-\log \mu(B(k)) + \sum_{r\in R} \sum_{m\in\Lambda(a)} \varphi((r+m)x_k) \right] \\
&\qquad = \frac{1}{|R|} \sum_k \mu(B(k)) \log\left(\exp\left(\sum_r \sum_m \varphi((r+m)x_k) \right) \Big/ \mu(B(k)) \right) \\
&\qquad \leq \frac{1}{|R|} \log \sum_k \exp \sum_r \sum_m \varphi((r+m)x_k) \\
&\qquad \leq \frac{1}{|R|} \log[(t+1)|\Lambda(a)|+1]^{|R|} \sum_{x\in E} \exp \sum_r \sum_m \varphi((r+m)x).
\end{aligned}
$$

If $\Lambda = \bigcup_{r\in R}(\Lambda(a)+r)$ then E is (δ, Λ)-separated; therefore

$$\frac{1}{|R|} \log \sum_{x\in E} \exp \sum_r \sum_m \varphi((r+m)x) \leq |\Lambda(a)| P(\varphi, \delta, \Lambda).$$

so that

$$s(\mu) + \mu(\varphi) - \epsilon \leq P(\varphi, \delta, \Lambda) + (1/|\Lambda(a)|)\log[(t+1)|\Lambda(a)|+1].$$

By taking $|\Lambda(a)|$ large then letting $\Lambda \uparrow \infty$, this yields

$$s(\mu) + \mu(\varphi) \leq P(\varphi). \tag{5.5}$$

We show now that equality holds in (5.5) for some μ. Let $\langle u\rangle = (2^u, \ldots, 2^u)$ and let μ be a limit of the sequence $\mu_{\varphi,\langle u\rangle}$. Choose now a partition of $\mathcal{A}$ consisting of sets with diameter $< \delta^*$, and with boundaries of μ-measure 0. Given $\epsilon > 0$, there exists u such that $s(\mu) + \epsilon/2 > (1/|\Lambda(\langle u\rangle)|)S(\mu, \mathcal{A}^{\Lambda(\langle u\rangle)})$ and since $\mu_{\varphi,\langle v\rangle}(A(k)) \to \mu(A(k))$ when $v \to \infty$, one can choose $V \geq u$ such that if $v \geq V$,

$$\begin{aligned} s(\mu) + \epsilon &> (1/|\Lambda(\langle u\rangle)|)S(\mu_{\varphi,\langle v\rangle}, \mathcal{A}^{\Lambda(\langle u\rangle)}) \\ &\geq (1/|\Lambda(\langle v\rangle)|)S(\mu_{\varphi,\langle v\rangle}, \mathcal{A}^{\Lambda(\langle v\rangle)}) \\ &\geq (1/|\Lambda(\langle v\rangle)|) \sum_{x\in\Pi_{\langle v\rangle}} \mu_{\varphi,\langle v\rangle}(\{x\})\log \mu_{\varphi,\langle v\rangle}(\{x\}) \end{aligned}$$

where we have used the subadditivity of $\Lambda \to S(\mu, \mathcal{A}^\Lambda)$, and then expansiveness. Using the definition of $\mu_{\varphi,\langle v\rangle}$ we obtain

$$\begin{aligned} s(\mu) + \epsilon &> -\frac{1}{|\Lambda(\langle v\rangle)|} \sum_{x\in\Pi_{\langle v\rangle}} \mu_{\varphi,\langle v\rangle}(\{x\})\Big[\sum_{m\in\Lambda(\langle v\rangle)} \varphi(mx) - \log Z(\varphi, \langle v\rangle)\Big] \\ &= -\mu_{\varphi,\langle v\rangle}(\varphi) + (1/|\Lambda(\langle v\rangle)|)\log Z(\varphi, \langle v\rangle) \end{aligned}$$

and the desired result follows by letting $\mu_{\varphi,\langle v\rangle} \to \mu$. We have thus proved (5.1).

Let $J_\varphi = \{\mu \in I\colon s(\mu) + \mu(\varphi) = P(\varphi)\}$; J_φ is the set where the affine upper semicontinuous function $\mu \to s(\mu) + \mu(\varphi)$ reaches its maximum; hence J_φ is convex and compact. If $\mu \in J_\varphi$, we have

$$\begin{aligned} P(\varphi + \psi) &\geq s(\mu) + \mu(\varphi + \psi) = s(\mu) + \mu(\varphi) + \mu(\psi) \\ &= P(\varphi) + \mu(\psi); \end{aligned}$$

hence $\mu \in I_\varphi$. Therefore $J_\varphi \subset I_\varphi$. If J_φ were different from I_φ one could find $\psi \in \mathcal{C}(\Omega)$ such that

$$\sup_{\mu\in I_\varphi} \mu(\psi) > \sup_{\mu\in J_\varphi} \mu(\psi). \tag{5.6}$$

Let $\mu_n \in J_{\varphi+\psi/n}$ and $\mu \in I_\varphi$, we have

$$\begin{aligned} \mu(\psi) &= n\mu(\psi/n) \leq n[P(\varphi + \psi/n) - P(\varphi)] \\ &\leq n[P(\varphi + \psi/n) - s(\mu_n) - \mu_n(\varphi)] \\ &= n[\mu_n(\varphi + \psi/n) - \mu_n(\varphi)] = \mu_n(\psi). \end{aligned}$$

If μ^* is a limit point of the sequence (μ_n), then $\mu^* \in J_\varphi$ (by upper semicontinuity of s), and therefore $\mu(\psi) \leq \mu^*(\psi)$ for all $\mu \in I_\varphi$, in contradiction with (5.6). We have thus shown that $J_\varphi = I_\varphi$.

We want now to prove (5.2). We already know by (5.5) that $s(\mu) \leq P(\varphi) - \mu(\varphi)$ and it remains to show that by proper choice of φ the right-hand side becomes as close as desired to $s(\mu)$. Let $C = \{(\mu, t) \in \mathcal{C}(\Omega)^* \times \mathbf{R}\colon \mu \in I$ and $0 \leq t \leq s(\mu)\}$. Since s is affine upper semicontinuous, C is convex and compact. Given $\mu^* \in I$ and $u > s(\mu^*)$ there exist (because C is convex and compact) $\varphi \in \mathcal{C}(\Omega)$ and $c \in \mathbf{R}$ such that

$$-\mu^*(\varphi) + c = u, \qquad -\mu(\varphi) + c > s(\mu), \quad \text{for all } \mu \in I;$$

hence $-\mu(\varphi) + u + \mu^*(\varphi) > s(\mu)$ and we have, if $\mu \in I_\varphi$,

$$\begin{aligned} 0 &\leq P(\varphi) - s(\mu^*) - \mu^*(\varphi) \\ &= s(\mu) + \mu(\varphi) - s(\mu^*) - \mu^*(\varphi) \\ &< u - s(\mu^*). \end{aligned}$$

The right-hand side is arbitrarily small and (5.2) follows.

5.2. **Remark.** If Ω is a basic set for an Axiom A diffeomorphism it is known [3] that $0 \in D$, i.e., the maximum of $s(\mu)$ is reached for just one $\mu \in I$. Further results on D have been obtained for Anosov diffeomorphisms using methods of statistical mechanics [18].

6. **The sets of invariant states.** In this section we study the set I of all $\mathbf{Z}^\nu$-invariant probability measures and its relations with the I_φ.

6.1. **Proposition.** *For each $\varphi \in \mathcal{C}(\Omega)$, I_φ is a Choquet simplex, and a face* (*see* [4]) *of the simplex I.*

It is well known that the set I of invariant probability measures is a simplex.(6) If $\mu \in I_\varphi$, let m_μ be the unique probability measure on I, carried by the extremal points of I, and with resultant μ. Writing $\hat{\varphi}(\nu) = \nu(\varphi)$, we have (see [4])

$$m_\mu(s + \hat{\varphi}) = s(\mu) + \mu(\varphi) = P(\varphi);$$

hence the support of m_μ is contained in $\{\nu \in I: s(\nu) + \nu(\varphi) = P(\varphi)\} = I_\varphi$. This shows that I_φ is a simplex and a face of I.

6.2. **Proposition.** *Suppose that $\mathcal{B}$ is dense in $\mathcal{C}(\Omega)$ and is a separable Banach space with respect to a norm $|||\cdot||| \geq \|\cdot\|$. If $\varphi \in \mathcal{B}$, then I_φ is the closed convex hull of the set of μ such that*

$$\mu = \lim_{n\to\infty} \mu_{\varphi(n)}, \qquad \lim_{n\to\infty} |||\varphi(n) - \varphi||| = 0, \qquad \varphi(n) \in D \cap \mathcal{B},$$

where D is defined in Theorem 3.2(a). *This applies in particular with $\mathcal{B} = \mathcal{C}(\Omega)$.*

We have $P(\varphi(n) + \psi) \geq P(\varphi(n)) + \mu_{\varphi(n)}(\psi)$ for all ψ, hence $P(\varphi + \psi) \geq P(\varphi) + \mu(\psi)$ so that $\mu \in I_\varphi$ if μ is of the above form.

Suppose now that I_φ were not in the closed convex hull of those μ. There would then exist $\psi \in \mathcal{B}$ such that

$$\sup_{\nu \in I_\varphi} \nu(\psi) > \sup_{\mu} \mu(\psi). \tag{6.1}$$

Let $\varphi(n) = \varphi + \psi/n + \chi_n \in D \cap \mathcal{B}$; then, by convexity of P, if $\nu \in I_\varphi$,

$$\nu(\psi/n + \chi_n) \leq \mu_{\varphi(n)}(\psi/n + \chi_n).$$

(6) See for instance Jacobs [10, p. 162].

Using Theorem 3.2(e) we may take $|||\chi_n||| < 1/n^2$; we have thus

$$\nu(\psi) - 1/n \leq \mu_{\varphi(n)}(\psi) + 1/n,$$

and if μ^* is a limit point of $(\mu_{\varphi(n)})$, $\nu(\psi) \leq \mu^*(\psi)$ in contradiction with (6.1).

6.3. **Proposition.** *The set of measures* μ *on* Ω *such that*

$$\mu(\varphi) \leq P(\varphi) \quad \textit{for all } \varphi \in \mathcal{C}(\Omega) \tag{6.2}$$

is I.

If $\mu \in I$ we have $\mu(\varphi) \leq P(\varphi) - s(\mu) \leq P(\varphi)$ because $s \geq 0$. Let now (6.2) hold for some $\mu \in \mathcal{C}(\Omega)^*$. By (2.8) we have

$$\mu(\varphi) - \mu(\tau_m \varphi) = t^{-1}\mu(t\varphi - t\tau_m \varphi) \leq t^{-1}P(t\varphi - t\tau_m\varphi) = t^{-1}P(0).$$

Letting $t \to \infty$ gives $\mu(\varphi) - \mu(\tau_m\varphi) \leq 0$. Replacing φ by $-\varphi$ yields $\mu(\varphi) = \mu(\tau_m\varphi)$. Therefore μ is $\mathbf{Z}^\nu$ invariant. Using now (2.7) and (2.8) we find

$$\pm\mu(\varphi) = \lim_{t\to\pm\infty} \frac{1}{|t|}\mu(t\varphi) \leq \lim_{t\to\pm\infty} \frac{1}{|t|}P(t\varphi)$$
$$\leq \lim_{t\to\pm\infty} \frac{1}{|t|}[P(0) + \|t\varphi\|] = \|\varphi\|$$

so that $\|\mu\| \leq 1$. Furthermore (2.8) shows that, for all t, $t\mu(1) = \mu(t) \leq P(0) + t$, so that $\mu(1) = 1$. Since $\|\mu\| = 1$ and $\mu(1) = 1$, μ is a probability measure.

6.4. **Proposition.**(7) *The set*

$$\mathcal{M}_p = \cup_a \left\{ \frac{1}{|\Lambda(a)|} \sum_{m\in\Lambda(a)} \delta_{mx} : x \in \Pi_a \right\}$$

is dense in I.

A vague neighbourhood of $\mu \in I$ is given by $\{\nu \in I: \|\nu - \mu\|_{\varphi_i} < \epsilon$ for $i = 1, \ldots, n\}$ where $\|\nu - \mu\|_{\varphi_i} = |\nu(\varphi_i) - \mu(\varphi_i)|$ and $\varphi_1, \ldots, \varphi_n \in \mathcal{C}(\Omega)$, $\epsilon > 0$. We assume without loss of generality that $\|\varphi_i\| \leq 1$ for $i = 1, \ldots, n$.

Given $\epsilon > 0$, we choose $\delta > 0$ such that $d(x,y) < \delta$ implies $|\varphi_i(x) - \varphi_i(y)| < \epsilon$ for $i = 1, \ldots, n$.

Let $p(\delta)$ be given by 1.2, $N > p(\delta)/\epsilon$ and $a = (N, \ldots, N)$, $b = (N + p(\delta), \ldots, N + p(\delta))$. By the density of measures with finite support we can choose $c_\alpha > 0$, $x_\alpha \in \Omega$ such that

$$\sum_\alpha c_\alpha = 1, \qquad \left\|\sum_\alpha c_\alpha \delta_{x_\alpha} - \mu\right\|_{\tau_m\varphi_i} < \epsilon,$$

(7) Sigmund [15] has proved this result by a somewhat different method for $\nu = 1$.

for $i = 1, \ldots, n$, and $m \in \Lambda(b)$. We have thus

$$\Big\| \sum_\alpha c_\alpha \delta_{mx_\alpha} - \mu \Big\|_{\varphi_i} < \epsilon \quad \text{for } m \in \Lambda(b);$$

hence

$$\Big\| \frac{1}{|\Lambda(b)|} \sum_{m\in\Lambda(b)} \sum_\alpha c_\alpha \delta_{mx_\alpha} - \mu \Big\|_{\varphi_i} < \epsilon. \tag{6.3}$$

By 1.3, we can choose $y_\alpha \in \Pi_b$ such that $|\varphi_i(mx_\alpha) - \varphi_i(my_\alpha)| < \epsilon$ for $m \in \Lambda(a)$, and we have $|\varphi_i(mx_\alpha) - \varphi_i(my_\alpha)| \leq 2$ for $m \in \Lambda(b)\backslash\Lambda(a)$; hence

$$\begin{aligned} &\Big\| \sum_{m\in\Lambda(b)} \sum_\alpha \frac{c_\alpha}{|\Lambda(b)|} \delta_{my_\alpha} - \sum_{m\in\Lambda(b)} \sum_\alpha \frac{c_\alpha}{|\Lambda(b)|} \delta_{mx_\alpha} \Big\|_{\varphi_i} \\ &\quad < \epsilon \frac{|\Lambda(a)|}{|\Lambda(b)|} + 2\frac{|\Lambda(b)| - |\Lambda(a)|}{|\Lambda(b)|} < \epsilon + 2(1+\epsilon)^\nu - 2. \end{aligned} \tag{6.4}$$

We can now find integers P, $M_\alpha > 0$ such that $\sum_\alpha M_\alpha = P^\nu$ and

$$\Big\| \sum_\alpha \frac{M_\alpha}{|\Lambda(b)|P^\nu} \sum_{m\in\Lambda(b)} \delta_{my_\alpha} - \sum_{m\in\Lambda(b)} \sum_\alpha \frac{c_\alpha}{|\Lambda(b)|} \delta_{my_\alpha} \Big\|_{\varphi_i} < \epsilon. \tag{6.5}$$

Let $c = ((N + p(\delta))P, \ldots, (N + p(\delta))P)$. By application of (1.3), there exists $y \in \Pi_c$ such that when $\tilde{m}$ varies over $\Lambda(c)$, my takes M_α times a value close to my_α for each α and each $m \in \Lambda(a)$. Close means $d(\tilde{m}y, my_\alpha) < \delta$. Then

$$\begin{aligned} &\Big\| \frac{1}{|\Lambda(c)|} \sum_{\tilde{m}\in\Lambda(c)} \delta_{\tilde{m}y} - \frac{1}{|\Lambda(b)|P^\nu} \sum_\alpha M_\alpha \sum_{m\in\Lambda(b)} \delta_{my_\alpha} \Big\|_{\varphi_i} \\ &\quad \leq \epsilon \frac{|\Lambda(a)|}{|\Lambda(b)|} + 2\frac{|\Lambda(b)| - |\Lambda(a)|}{|\Lambda(b)|} < \epsilon + 2(1+\epsilon)^\nu - 2. \end{aligned} \tag{6.6}$$

Finally, (6.3), (6.4), (6.5), (6.6) give

$$\Big\| \frac{1}{|\Lambda(c)|} \sum_{\tilde{m}\in\Lambda(c)} \delta_{\tilde{m}y} - \mu \Big\|_{\varphi_i} < 4\epsilon + 4(1+\epsilon)^\nu - 4,$$

proving the proposition.

6.5. **Proposition.**[8] (a) *The set of ergodic measures (extremal points of I) is residual in I.*

(b) *The set of measures with zero entropy is residual in I.*

Since $\mathcal{M}_p$ is dense (Proposition 6.4) and consists of ergodic measures with zero entropy, it suffices to show that the set of ergodic measures and the set of measures with zero entropy are G_δ (i.e. countable intersections of open sets). For ergodic measures this is well known (see [4]); for measures with zero entropy, it follows from the fact that the entropy is upper semicontinuous.

[8] See Sigmund [15] where other residual sets are also discussed.

Added in proof. A proof of the variational principle (0, 1) has been obtained without the expansiveness and specification assumptions by P. Walters (preprint).

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INSTITUT DES HAUTES ETUDES SCIENTIFIQUES, BURES-SUR-YVETTE, FRANCE